

# Gauge-Invariant Perturbations of Varying-Alpha Cosmologies

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## Abstract

Using a gauge-invariant formalism we derive and solve the perturbed cosmological equations for the BSBM theory of varying fine structure 'constant'. We calculate the time evolution of inhomogeneous perturbations of the fine structure constant,  $\frac{\delta\alpha}{\alpha}$  on small and large scales with respect to the Hubble radius. In a radiation-dominated universe small inhomogeneities in  $\alpha$  will decrease on large scales but on scales smaller than the Hubble radius they will undergo stable oscillations. In a dust-dominated universe small inhomogeneous perturbations in  $\alpha$  will become constant on large scales and on small scales they will increase as  $t^{2/3}$ , and  $\frac{\delta\alpha}{\alpha}$  will track  $\frac{\delta\rho_m}{\rho_m}$ . If the expansion accelerates, as in the case of a  $\Lambda$  or quintessence-dominated phase, inhomogeneities in  $\alpha$  will decrease on both large and small scales. The amplitude of perturbations in  $\alpha$  will be much smaller than that of matter or radiation perturbations. We also present a numerical study of the non-linear evolution of spherical inhomogeneities in radiation and dust universes by means of a comparison between the evolution of flat and closed Friedmann models with time-varying  $\alpha$ . Various limitations of these simple models are also discussed.

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# 1 Introduction

A third sample of observations of quasar absorption-line spectra has been found to be consistent with a time variation in the value of the fine structure 'constant' between redshifts  $z = 0.2 - 3.7$  and the present. The entire data set of 128 objects gives spectra consistent with a shift of  $\Delta\alpha/\alpha \equiv [\alpha(z) - \alpha_0]/\alpha_0 = -0.57 \pm 0.10 \times 10^{-5}$ , where  $\alpha_0$  is the present value of the fine structure constant [1, 2, 3, 4]. Extensive analysis has yet to find a selection effect that can explain the sense and magnitude of the relativistic line-shifts underpinning these deductions. Motivated by these observations, there has been considerable theoretical investigation of the cosmological consequences of varying  $\alpha$ , [5]. Barrow, Sandvik, and Magueijo [6, 7, 8, 9, 10] have studied in detail the cosmological consequence of an extension of the varying- $\alpha$  Maxwell equations formulated by Bekenstein [11]. We call these BSBM theories. They allow us to understand the effects of the expansion of the universe on variations of  $\alpha$  and to evaluate the effects of varying  $\alpha$  on free fall which leads to potentially observable violations of the weak equivalence principle [12, 13, 14, 15]. They also allow us to investigate whether or not other cosmological observations are consistent with the small variations in  $\alpha$  that are required to fit the quasar observations. Other theories, founded explicitly upon variation in the velocity of light, have also been proposed but their main interest is in connection with events in the very early universe [16, 17, 18, 19] and the problem of whether or not non-inflationary explanations for particular features of the large-scale structure of the universe are possible [20]. Others have begun to consider the implications for grand unification of coupling strengths, [21, 15, 22, 23, 24, 25, 26, 27, 42, 43], and astronomical probes of the constancy of the electron-proton mass ratio have reported possible evidence for time variation, [28], but as yet the statistical significance is low.

Barrow, Sandvik, and Magueijo [6, 7, 8, 9] have shown that BSBM theories have a number of appealing properties. They predict that there should be no variation of  $\alpha$  during the radiation era in our universe and none during any present or late-time curvature or cosmological constant dominated era. During the dust era  $\alpha$  should grow (leading to  $\Delta\alpha/\alpha < 0$  as observed) but only as  $\log(t)$ . This behaviour allows the quasar data to be accommodated without producing conflict with recent geonuclear limits on allowed variations of  $\alpha$ , like the Oklo natural reactor limits of  $\Delta\alpha/\alpha \lesssim 10^{-7}$  [29, 30, 31, 32, 34] because they are imposed at a very low effective redshift of  $z \approx 0.15$ , at which

time the universe has begun accelerating and all variations in  $\alpha$  are damped out. Recent deductions of a possible upper limit of  $\Delta\alpha/\alpha \lesssim 3 \times 10^{-7}$  at  $z = 0.45$  from nuclear  $\beta$  decays are potentially more restrictive [35, 36, 37]. However, it must be remembered that both these nuclear limits are derived from a local solar-system environment. In the absence of a theory relating the value of, and rate of change of,  $\alpha$  on the cosmological scales where quasar lines form to their values in the virialised local inhomogeneities where galaxies, stars and planets form, one should be wary of ignoring the possible corrections that must be introduced when comparing planetary and quasar bounds: the density of the Earth would not, for example, be a good indicator of the density of the universe. Thus, inhomogeneity is an important factor in the study of varying-constant cosmological theories. In this paper we are going to study the evolution of small, gauge-invariant perturbations to the exact Friedmann-Robertson-Walker solutions of the BSBM theory. The results of this investigation will reveal whether one must worry about fast growth of small initial inhomogeneities in the value of  $\alpha$ , which would lead to spatial variations in the value of  $\alpha$  that might be more significant than the time variations at late times.

Bekenstein [11] generalised Maxwell's equations to include varying  $e$  and this theory was then generalised by Sandvik, Barrow and Magueijo [6] to include gravitation. In the BSBM varying- $\alpha$  theory, the quantities  $c$  and  $\hbar$  are taken to be constant, while  $e$  varies as a function of a real scalar field  $\psi$ , with  $\alpha = e^2$ , hence

$$e = e_0 \exp[\psi] \tag{1}$$

$$\alpha = \alpha_0 \exp[2\psi] \tag{2}$$

Our discussion is organised as follows. In section 2 of the paper we give the gauge-invariant perturbation equations following the development used in general relativity. In section 3 we specify the BSBM cosmology with varying  $\alpha$  and derive the gauge invariant linear perturbation equations which couple the perturbations in the gravitational field to those in  $\alpha$  and the density. In section 4 we solve for the time-evolution of small inhomogeneities in the fine structure 'constant' on large and small scales for radiation, dust, and cosmological constant dominated expansion of the background universe. In section 5 we extend these studies into the non-linear regime by means of numerical solutions for flat and closed FRW universes. The evolution of spherical curvature inhomogeneities in density and in  $\alpha$  is followed by computing the

difference in time evolution between the FRW models of different curvature. These studies also reveal for the first time the behaviour of  $\alpha$  in closed FRW models in the BSBM theory. These results are discussed and conclusions drawn in section 6.

Units will be used in which  $c = \hbar = 1$ ; Greek indices run from 0 to 3 and Latin indices only over the spatial degrees of freedom 1 to 3. The Einstein summation convention is assumed;  $a(t)$  is the scale factor of the background Friedmann-Robertson-Walker (FRW) metric and  $G$  is Newton's gravitation constant.

## 2 The Background and Perturbations

In what follows we shall assume that the metric of space-time deviates only by a small amount from a homogeneous, isotropic FRW space-time which is defined to be the background universe. In this case, it is convenient to split the metric into two parts: the background metric, and its perturbation. We observe that the universe is nearly homogeneous and isotropic on large scales (metric perturbations to the cosmic microwave background radiation are observed to be small  $\approx 10^{-5} \ll 1$ ).

The background line element is:

$$ds^2 = {}^{(0)}g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j), \quad (3)$$

where  $\eta$  is the conformal time,

$$d\eta = a^{-1}dt.$$

We choose the background metric to be the FRW metric, so

$$\gamma_{ij} = \delta_{ij}[1 + \kappa(x^2 + y^2 + z^2)]^{-2},$$

where  $\kappa = 0, 1, -1$  depending on whether the three-dimensional hypersurfaces of constant  $\eta$  are flat, closed or open. The Einstein equations are:

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 8\pi GT^\mu_\nu, \quad (4)$$

where  $R^\mu_\nu$  is the Ricci tensor,  $R \equiv R^\mu_\mu$  is the Ricci scalar and  $T^\mu_\nu$  is the total energy-momentum tensor. We shall exploit the fact that the BSBM

theory for varying  $\alpha$  can be expressed as general relativity with a particular linear combination of energy-momentum tensors. For the moment consider the presence of a single energy-momentum tensor. In the next section this will be decomposed appropriately.

For the background metric in equation (3), in conformal time, the Einstein equations reduce to the  $0 - 0$  equation

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2T_0^0 - \kappa, \quad (5)$$

where  $\mathcal{H} \equiv \frac{a'}{a}$  using conformal time, and the  $i - i$  equation:

$$\mathcal{H}' + \mathcal{H}^2 = \frac{4\pi G}{3}a^2T - \kappa, \quad T \equiv T_\mu^\mu \quad (6)$$

For the background metric (3), the space-space part of the Ricci tensor  $R_j^i$  is proportional to  $\delta_j^i$ . Thus, for an isotropic background universe, the energy-momentum tensor must also be spatially diagonal,  $T_j^i \propto \delta_j^i$ , in order that the Einstein equations are satisfied. Differentiating (5) with respect to  $\eta$  and subtracting  $2a'$  we get the continuity equation for matter  $\nabla_\mu T_0^\mu = 0$  :

$$dT_0^0 = -(4T_0^0 - T)d \ln a. \quad (7)$$

We now introduce small perturbations around the FRW background and follow the gauge-invariant approach of Mukhanov [38]. The full line element may be expressed as:

$$ds^2 = {}^{(0)}g_{\mu\nu}dx^\mu dx^\nu + \delta g_{\mu\nu}dx^\mu dx^\nu, \quad (8)$$

where  $\delta g_{\mu\nu}$  describes the perturbation. The full metric has been decoupled into its background parts and perturbation parts:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta g_{\mu\nu}. \quad (9)$$

In general, the metric perturbations can be divided into three distinct types: scalar, vector and tensor. Neither of the vector and tensor perturbations exhibit growing instabilities in dust and radiation universes. Vector perturbations decay kinematically in an expanding universe whereas tensor perturbations lead to gravitational waves that do not couple to the isotropic

energy-density and pressure inhomogeneities. However, scalar perturbations may lead to growing inhomogeneities which, in turn, have an important effect on the dynamics of matter and thereby on the time and space variations of the fine structure constant in the BSBM theory. In the linear approximation, scalar, vector and tensor perturbations evolve independently and can be considered separately. In this paper we will consider only the scalar perturbation modes.

The most general form of the scalar metric perturbations is constructed using four scalar quantities which are functions of space and time coordinates:

$$\delta g_{\mu\nu} = \begin{pmatrix} 2\phi & -B_{|i} \\ -B_{|i} & 2(\chi\gamma_{ij} - E_{|ij}) \end{pmatrix},$$

where  $|i$  represents the three-dimensional covariant derivative.

From the above equation and eqn. (9), the line element for the background and for the scalar metric perturbations is

$$ds^2 = a^2(\eta)\{(1 + 2\phi)d\eta^2 - 2B_{|i}dx^i d\eta - [(1 - 2\chi)\gamma_{ij} + 2E_{|ij}]dx^i dx^j\}. \quad (10)$$

## 2.1 Gauge-invariant Variables

Gauge-invariant variables are unchanged under all infinitesimal scalar coordinate transformations, so they are independent of the background coordinates. Such quantities can be constructed out of the four scalar functions  $\Phi$ ,  $\Psi$ ,  $E$  and  $B$  [39]. The simplest gauge-invariant linear combinations which span the space of gauge-invariant variables that can be constructed from the metric variables alone are:

$$\Phi = \phi + \frac{[(B - E')a]'}{a}, \quad \Psi = \chi - \frac{a}{a'}(B - E'). \quad (11)$$

In general, a scalar quantity  $f(\eta, \mathbf{x})$  defined in the spacetime can be split into its background value and a perturbation  $f(\eta, \mathbf{x}) = f_0(\eta) + \delta f(\eta, \mathbf{x})$ . Since, in general,  $\delta f$  is not gauge invariant, we cannot use this scalar quantity without modification if we want to have gauge-invariant equations. Hence, we consider the following gauge-invariant combination:

$$\delta f^{(gi)} = \delta f + f'_0(B - E'). \quad (12)$$

The freedom of gauge choice can be used to impose two conditions on the four scalar functions. The *longitudinal gauge* is defined by the conditions  $E = B = 0$ . This gauge choice has the advantage of ruling out the complications of residual gauge modes. Also, in this gauge  $\phi$  and  $\chi$  coincide with the gauge-invariant variables  $\Phi$  and  $\Psi$  respectively. In this longitudinal gauge, the metric takes the form:

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)\gamma_{ij}dx^i dx^j],$$

and the gauge invariants  $\Phi$  and  $\Psi$  become the amplitudes of the metric perturbations in the longitudinal coordinate system. In the case where there are no space-space components in the energy-momentum tensor, so  $T_j^i \propto \delta_j^i$ , we have that  $\Phi = \Psi$  and there remains only one free metric perturbation variable which is a generalisation of the Newtonian gravitational potential.

### 3 Linear Theory for Cosmological Perturbations

For small perturbations of the metric, the Einstein tensor can be written as  $G_\nu^\mu = {}^{(0)}G_\nu^\mu + \delta G_\nu^\mu$ , and the energy-momentum tensor can be split in a similar way. The fully perturbed Einstein equations can be obtained for scalar perturbation modes with the line element given by (10). However, these equations are not gauge invariant, since they contain non gauge-invariant quantities. In order to have gauge-invariant equations we need to replace  $\phi$  and  $\chi$  by the gauge-invariant variables  $\Phi$ ,  $\Psi$  and  $B - E'$  and to construct the gauge-invariant equivalents of  $\delta G_\nu^\mu$  and  $\delta T_\nu^\mu$  we need to rewrite them as [38]:

$$\begin{aligned} \delta G_0^{(gi) \ 0} &= \delta G_0^0 + ({}^{(0)}G_0^0)'(B - E'), \\ \delta G_j^{(gi) \ i} &= \delta G_j^i + ({}^{(0)}G_j^i)'(B - E'), \\ \delta G_i^{(gi) \ 0} &= \delta G_i^0 + ({}^{(0)}G_0^0 - \frac{1}{3} {}^{(0)}G_k^k)(B - E')|_i, \end{aligned} \tag{13}$$

and analogously for  $\delta T_\nu^\mu$ ,

$$\begin{aligned} \delta T_0^{(gi) \ 0} &= \delta T_0^0 + ({}^{(0)}T_0^0)'(B - E'), \\ \delta T_j^{(gi) \ i} &= \delta T_j^i + ({}^{(0)}T_j^i)'(B - E'), \end{aligned}$$

$$\delta T_i^{(gi)0} = \delta T_i^0 + ({}^{(0)}T_0^0 - \frac{1}{3} {}^{(0)}T_k^k)(B - E')|_i.$$

The components of the perturbed Einstein equations linearised around small perturbations of the background are  $\delta G_\nu^{(gi)\mu} = 8\pi G \delta T_\nu^{(gi)\mu}$ :

$$\begin{aligned} \delta G_0^0 &= \nabla^2 \Phi - 3\mathcal{H}\Phi' - 3\Phi(\mathcal{H}^2 - \kappa) = 4\pi G a^2 \delta T_0^{(gi)0} \\ \delta G_i^0 &= \partial_i(a\Phi)' = 4\pi G a \delta T_i^{(gi)0} \\ \delta G_i^j &= \Phi'' + 3\mathcal{H}\Phi' + \Phi(2\mathcal{H}' + \mathcal{H}^2 - \kappa) = -4\pi G a^2 \delta T_j^{(gi)i} \end{aligned} \quad (14)$$

where we have already simplified the equations since the energy-momentum tensor that we will be considering in section 3 has no space-space components and so  $\Phi = \Psi$ .

In order to close our system of equations, we need equations of motion for the matter formulated in a gauge-invariant way. This requires explicit expressions for the energy-momentum tensor and so we must now specify the BSBM theory.

### 3.1 The Model and the Background Equations

#### 3.1.1 The BSBM Theory

The action for the universe in the BSBM theory is given by:

$$S = \int d^4x \sqrt{-g} (\mathcal{L}_{grav} + \mathcal{L}_{matter} + \mathcal{L}_\psi + \mathcal{L}_{em} e^{-2\psi}), \quad (15)$$

where  $\mathcal{L}_\psi = \frac{\omega}{2} \partial_\mu \psi \partial^\mu \psi$ ,  $\omega$  is a coupling constant,  $\mathcal{L}_{em} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}$ , and  $\psi$  was defined in eqn. (1). The gravitational Lagrangian is the usual  $\mathcal{L}_g = -\frac{1}{16\pi G} R$ , with  $R$  the curvature scalar, and we have defined an auxiliary gauge potential  $a_\mu = \epsilon A_\mu$ , where  $\varepsilon(x^\mu) = e/e_0$  describes change in the electron charge away from a constant reference value  $e_0$ . The field tensor  $f_{\mu\nu} = \epsilon F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ , so the covariant derivative takes the usual form,  $D_\mu = \partial_\mu + ie_0 a_\mu$ . The dependence on  $\epsilon$  in the Lagrangian then occurs only in the kinetic term for  $\epsilon$  and through the  $F^2 = f^2/\epsilon^2$  term.

It was shown in [10] and [9] that, in the context of the BSBM model, the homogeneous evolution of  $\psi$  does not create significant metric perturbations at late times and the cosmological time-evolution of the expansion scale-factor is very well approximated by the usual power-laws found in  $\kappa = 0$  FRW models filled with a perfect fluid.



Therefore, we will assume there are no major modifications in the perturbed spacetime which would lead to changes of the behaviour of the perturbed variables of a perturbed FRW spacetime filled with perfect fluid. That is, we will assume that the energy-density perturbations and the metric potential are the same as in a FRW universe with no variation of  $\alpha$ . Physically, this is to be expected for most of the evolution, although this assumption might break down (along with much else) on approach to initial and final cosmological singularities. It is a reflection of the fact that the changes in  $\alpha$  have negligible feedback into the changes in the expansion, which are governed to leading order by gravity. The principal effects are those of perturbations in the matter density and expansion rate on the evolution of  $\alpha$ . This simplification will allow us to write the time and space variations of the scalar field,  $\delta\psi$ , as a functions of  $\rho_m$ ,  $\rho_r$ ,  $\delta\rho_m$ ,  $\delta\rho_r$ ,  $\Phi$  and  $\psi$ , where  $\delta\rho_m$ ,  $\delta\rho_r$  and  $\Phi$  will be given by the solutions found in ref. [38] for universes with no variation of  $\alpha$ ; the field  $\psi$  will be given by the solutions found previously in [10]. In order to find an expression for  $\delta\psi$  in terms of these quantities we need to write the gauge-invariant linearly perturbed Einstein equations for the BSBM model.

### 3.1.2 The Background Equations

We vary the action (15) with respect to the metric to obtain the generalised Einstein equations:

$$G_{\nu}^{\mu} = 8\pi G (T_{\nu}^{matter\ \mu} + T_{\nu}^{\psi\ \mu} + T_{\nu}^{em\ \mu})$$

where  $T_{\mu\nu}^{mat} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_{mat})}{\delta g^{\mu\nu}}$  is the energy-momentum tensor for perfect-fluid matter fields, and

$$T_{\mu\nu}^{matter} = (\rho_m + p_m)u_{\mu}u_{\nu} - p_m g_{\mu\nu},$$

where  $u^{\mu} = \delta_0^{\mu}$  is the comoving fluid 4-velocity;  $T_{\nu}^{\psi\ \mu}$  and  $T_{\nu}^{em\ \mu}$  are the energy-momentum tensors for the kinetic energy of the field  $\psi$  and the electromagnetic field respectively:

$$T_{\mu\nu}^{\psi} = \omega \partial_{\mu}\psi \partial_{\nu}\psi - \frac{\omega}{2} g_{\mu\nu} \partial_{\beta}\psi \partial^{\beta}\psi, \quad T_{\mu\nu}^{em} = F_{\mu\beta} F_{\nu}^{\beta} e^{-2\psi} - \frac{1}{4} g_{\mu\nu} F_{\sigma\beta} F^{\sigma\beta} e^{-2\psi}.$$

Note, the total energy density of the electromagnetic field is the sum of the Coulomb energy density  $\zeta\rho_m$  and the radiation energy density  $\rho_r$ , where

$-1 \leq \zeta \leq 1$  is the fraction of mass density  $\rho_m$  of matter in the form of the Coulomb energy. We will then consider  $T_{\nu}^{em \mu}$  as a perfect fluid:

$$T_{\mu\nu}^{em} = (|\zeta|\rho_m + \rho_r + p_m + p_r)e^{-2\psi}u_\mu u_\nu - (p_m + p_r)e^{-2\psi}g_{\mu\nu}$$

The propagation equation for  $\psi$  comes from the variational principle as:

$$\partial_\mu [\sqrt{-g}g^{\mu\nu}\partial_\nu\psi] = -\frac{2}{\omega}\sqrt{-g}e^{-2\psi}\mathcal{L}_{em}. \quad (16)$$

This equation determines how  $e$ , and hence  $\alpha$ , varies with time. It is clear that  $\mathcal{L}_{em}$  vanishes for a sea of pure radiation because  $\mathcal{L}_{em} = (E^2 - B^2)/2 = 0$ . In order to make quantitative predictions we need to know how non-relativistic matter contributes to the right hand side for equation (16), through  $\mathcal{L}_{em} = \zeta\rho_m$ .

The background equations can now be explicitly obtained:

$$\begin{aligned} 3\mathcal{H}^2 &= 8\pi G a^2 \left( \rho_m + (\rho_r + |\zeta|\rho_m)e^{-2\psi} + \frac{\omega}{2}a^{-2}\psi'^2 \right) + a^2\Lambda - 3\kappa \quad (17) \\ 3\mathcal{H}' &= -4\pi G a^2 \left( \rho_m (1 + |\zeta|e^{-2\psi}) + 2\rho_r e^{-2\psi} + 2\omega a^{-2}\psi'^2 \right) + a^2\Lambda \end{aligned}$$

where  $\Lambda$  is the cosmological constant. The equation of motion for the  $\psi$  field is:

$$2\mathcal{H}\psi' + \psi'' = \frac{2|\zeta|a^2}{\omega}\rho_m e^{-2\psi} \quad (18)$$

The conservation equations for the matter fields,  $\rho_r$  and  $\rho_m$  respectively, are:

$$\rho_m' + 3\mathcal{H}\rho_m = 0 \quad (19)$$

$$\rho_r' + 4\mathcal{H}\rho_r' = 2\psi'\rho_r. \quad (20)$$

## 3.2 Linear Perturbations

The perturbed components of the total energy-momentum ( $T_{\mu\nu}^{total} = T_{\mu\nu}^{mat} + T_{\mu\nu}^{\psi} + T_{\mu\nu}^{em}$ ) arise from perturbations of the different matter fields which are time and space dependent. In particular, we have  $\rho_m \rightarrow \rho_m + \delta\rho_m$ ,  $\rho_r \rightarrow \rho_r + \delta\rho_r$  and  $\psi \rightarrow \psi + \delta\psi$ . Note also that we have to perturb the fluid 4-velocity field, so we have  $u_i \rightarrow u_i + \delta u_i$ , where  $i = 1, 2, 3$ .

In order to have gauge-invariant equations we need to express the perturbed energy-momentum tensor in terms of the gauge-invariant energy density, pressure, scalar field and velocity field perturbations. The gauge invariants  $\delta\rho_m^{(gi)}$ ,  $\delta\rho_r^{(gi)}$  and  $\delta\psi^{(gi)}$  are defined in the same way as the gauge-invariant perturbation of a general four-scalar, see equation (11), so:

$$\delta\rho_m^{(gi)} = \delta\rho_m + \rho'_m(B-E'), \quad \delta p^{(gi)} = \delta p + p'(B-E'), \quad \delta\psi^{(gi)} = \delta\psi + \psi'(B-E'),$$

where  $\delta p$  is the perturbed pressure for a specific component and the gauge-invariant three-velocity  $\delta u_i^{(gi)}$  is given by [38]:

$$\delta u_i^{(gi)} = \delta u_i + a(B-E')|_i.$$

The physical meaning of the quantities which enter the gauge-invariant equations is very simple: they coincide with the corresponding perturbations in the longitudinal gauge. From now on we will drop the superscript  $(gi)$  since we will always be dealing only with gauge-invariant quantities.

We can now write the gauge-invariant linearly perturbed energy-momentum tensor:

$$\delta T_0^{matter\ 0} = \delta\rho_m, \quad (21)$$

$$\delta T_i^{matter\ 0} = (\rho_m + p_m) a^{-1} \delta u_i, \quad (22)$$

$$\delta T_j^{matter\ i} = -\delta p_m \delta_j^i, \quad (23)$$

$$\delta T_0^{\psi\ 0} = \omega a^{-2} (\psi' \delta\psi' - \psi'^2 \Phi), \quad (24)$$

$$\delta T_i^{\psi\ 0} = \omega a^{-2} \psi' \partial_i \delta\psi, \quad (25)$$

$$\delta T_j^{\psi\ i} = \omega a^{-2} (\psi'^2 \Phi - \psi' \delta\psi') \delta_j^i, \quad (26)$$

$$\delta T_0^{em\ 0} = (|\zeta| \delta\rho_m + \delta\rho_r) e^{-2\psi} - 2e^{-2\psi} (|\zeta| \rho_m + \rho_r) \delta\psi, \quad (27)$$

$$\delta T_i^{em\ 0} = e^{-2\psi} (|\zeta| \rho_m + \rho_r + p_m + p_r) a^{-1} \delta u_i, \quad (28)$$

$$\delta T_j^{em\ i} = (2\delta\psi (p_m + p_r) e^{-2\psi} - (\delta p_m + \delta p_r) e^{-2\psi}) \delta_j^i. \quad (29)$$

We have assumed there are no anisotropic stresses in the energy-momentum tensors and we have considered only adiabatic perturbations; that is, we

consider the pressure perturbations to depend only on the energy-density perturbations. In the dust and radiation cases this means  $p_m = 0$ ,  $\delta p_m = 0$ , or  $p_r = \frac{1}{3}\rho_r$  and  $\delta p_r = \frac{1}{3}\delta\rho_r$ , respectively.

The fully perturbed gauge-invariant Einstein equations can be obtained from (14) using the expressions above for  $\delta T_\nu^\mu$  (21-27). We have

$$\begin{aligned} \Phi \left( -3\mathcal{H}^2 + 3k + 4G\pi\omega\psi'^2 \right) + \nabla^2\Phi - 4G\pi\omega\psi'\delta\psi' - 3\mathcal{H}\Phi' = \\ 4G\pi a^2 e^{-2\psi} \left[ (e^{2\psi} + |\zeta|) \delta\rho_m + \delta\rho_r - 2\delta\psi (\rho_r + |\zeta|\rho_m) \right] \end{aligned} \quad (30)$$

$$\mathcal{H}\nabla^2\Phi + \nabla^2\Phi' = 4G\pi\omega\psi'\nabla^2\delta\psi - \frac{4G\pi}{3} a e^{-2\psi} \left[ 4\rho_r + 3(e^{2\psi} + |\zeta|)\rho_m \right] \nabla\delta u \quad (31)$$

$$\begin{aligned} \Phi \left( \mathcal{H}^2 + 2\mathcal{H}' - k + 4G\pi\omega\psi'^2 \right) + 3\mathcal{H}\Phi' + \Phi'' = \\ \frac{4G\pi}{3} \left[ a^2 e^{-2\psi} (\delta\rho_r - 2\rho_r\delta\psi) + 3\omega\psi'\delta\psi' \right] \end{aligned} \quad (32)$$

It is useful to write the perturbed energy-momentum conservation equations for each component:

$$\delta\psi'' = \frac{2|\zeta|}{\omega} e^{-2\psi} a^2 [\delta\rho_m + 2\rho_m(\Phi - \delta\psi)] - 2\mathcal{H}\delta\psi' + \nabla^2\delta\psi + 4\psi'\Phi', \quad (33)$$

$$\delta\rho_r' = -\frac{2}{3} \left[ \delta\rho_r (6\mathcal{H} - 3\psi') + \rho_r (2\nabla\delta u - 3\delta\psi' - 6\Phi') \right], \quad (34)$$

$$\delta\rho_m' = -3\mathcal{H}\delta\rho_m - \rho_m (\nabla\delta u - 3\Phi'). \quad (35)$$

From these expressions it is clear that perturbations in  $\alpha$  are sourced by perturbations in the dust, but not vice versa. Hence we expect that, in a dust-dominated universe with varying  $\alpha$ , the cold dark matter perturbations will behave as in a dust-dominated universe with no varying  $\alpha$ . Notice however that the same cannot be concluded so easily for a radiation-dominated era, since there is a source term in equation (34) proportional to  $\delta\psi'$ . We expect this term to be negligible at large scales, but that might not be the case on small scales.

The gauge-invariant perturbation for  $\alpha$  is given by eqn. (1) as

$$\frac{\delta\alpha}{\alpha} = 2\delta\psi.$$

From equations (30), (31), (32), using (17) to simplify, we obtain the general form for  $\delta\psi$ , the perturbation to the scalar field which drives variations in the fine structure 'constant', as

$$\delta\psi = \frac{1}{8G\pi a^2 (2\rho_r + 3|\zeta|\rho_m)} \left( 4G\pi a^2 [2(\delta\rho_r - 4\rho_r\Phi) + 3(e^{2\psi} + |\zeta|)] \right. \\ \left. (\delta\rho_m - 2\rho_m\Phi)] + 3e^{2\psi} \left[ 2\Phi \left( 3\mathcal{H}^2 - k - 4G\pi\omega\psi'^2 \right) - \nabla^2\Phi + 6\mathcal{H}\Phi' + \Phi'' \right] \right) \quad (36)$$

This is a gauge-invariant expression for  $\delta\psi$ , written as a function of the gauge-invariant quantities  $\Phi$ ,  $\delta\rho_m$ , and  $\delta\rho_r$ .

## 4 Evolution of the Perturbations

In a spatially flat universe,  $\kappa = 0$ , the general gauge-invariant expression for  $\delta\psi$  becomes:

$$\delta\psi = \frac{1}{8G\pi a^2 (2\rho_r + 3|\zeta|\rho_m)} \left( 4G\pi a^2 [2(\delta\rho_r - 4\rho_r\Phi) + 3(e^{2\psi} + |\zeta|)] \right. \\ \left. (\delta\rho_m - 2\rho_m\Phi)] + 3e^{2\psi} \left[ \Phi \left( 6\mathcal{H}^2 - 8G\pi\omega\psi'^2 \right) - \nabla^2\Phi + 6\mathcal{H}\Phi' + \Phi'' \right] \right). \quad (37)$$

It was shown in [9] and [10] that the cosmological evolution of the metric scale factor is unchanged (up to very small logarithmic corrections) to leading order by the time evolution of  $\psi$ . The dominant effect is that of the evolution of the scale factor on the evolution of  $\psi$  through its propagation equation.

It is known that perturbations of massless scalar fields, or scalar fields with a very small mass are negligible with respect to perturbations in the matter fields and the gravitational potential. Guided by this, in order to obtain the evolutionary behaviour for  $\frac{\delta\alpha}{\alpha}$ , we will assume that the matter field perturbations,  $\delta\rho_m$  and  $\delta\rho_r$ , and the metric perturbations,  $\Phi$ , are unaffected by the  $\psi$  perturbations to leading order. We will assume that these three quantities will therefore be the same to this order as they are in a flat FRW universe filled with barotropic matter and a minimally coupled scalar field.

These assumptions are valid if the energy density of  $\psi$  is much smaller than the energy density of the matter fields, so  $\Phi$  will be driven only by the matter perturbations. If we examine the perturbations in the non-linear regime it is confirmed that  $\frac{\delta\psi}{\psi} \ll \frac{\delta\rho}{\rho}$  (see figure (4)).

## 4.1 Radiation-Dominated Universes

In any radiation-dominated era in which the expansion of the universe is dominated by relativistic particles with an equation of state  $p = \frac{1}{3}\rho$ , we can neglect the non-relativistic stresses in the universe, in particular, the cold dark matter and the cosmological constant since  $\rho_r \gg \rho_m \gg \rho_\Lambda$  to a good approximation. If we assume that  $\rho_\Lambda = \rho_\Lambda = 0 = \kappa$  and  $\delta\rho_m = 0$ , the background equations of motion give the usual conformal time evolution for the scale factor and the energy density of the radiation:

$$\rho_r = \rho_{r_0} a^{-4} e^{2\psi} \quad a = \sqrt{8\pi G \rho_{r_0}} \eta$$

where  $\rho_{r_0}$  is a constant.

The perturbations in the barotropic matter fluid and the potential  $\Phi$  come from the equations (30), (31). The usual flat FRW solutions with constant  $\alpha$  are obtained by setting the terms proportional to  $\psi$  to zero, (32). The general solution of these equations can be obtained by expanding the physical quantities in terms of the eigenfunctions of the operator  $\nabla^2$  (where  $-k^2$  denotes the eigenvalue of this operator) and solving for each mode separately. Resuming the terms, the general solutions of the linearly perturbed equations for the potential  $\Phi$  and the barotropic matter are:

$$\Phi = \eta^{-3} \{ [w\eta \cos(w\eta) - \sin(w\eta)] C_1 + [w\eta \sin(w\eta) + \cos(w\eta)] C_2 \} e^{i\mathbf{k}\mathbf{x}} \quad (38)$$

and

$$\begin{aligned} \frac{\delta\rho_r}{\rho_r} = & \frac{4}{\eta^3} \left( C_1 \{ \eta w \left[ 1 - \frac{1}{2} (\eta w)^2 \right] \cos(\eta w) + [(\eta w)^2 - 1] \sin(\eta w) \} \right. \\ & \left. + \{ [1 - (\eta w)^2] \cos(\eta w) + \eta w \left[ 1 - \frac{1}{2} (\eta w)^2 \right] \sin(\eta w) \} C_2 \right) e^{i\mathbf{k}\mathbf{x}}, \end{aligned} \quad (39)$$

where we have expanded the general solution in plane waves since we are assuming a spatially flat universe;  $C_1$  and  $C_2$  are arbitrary functions of the

spatial coordinates;  $k$  is the wave vector mode and  $w = k/\sqrt{3}$ . Note that these quantities are all expressed in their gauge-invariant format. Finally, to calculate the explicit time dependence of  $\delta\psi$ , we need to use the background solution for  $\psi$ :

$$\psi = \frac{1}{2} \log(8N) + \frac{1}{4} \log\left(\frac{a_0}{2}\eta^2\right). \quad (40)$$

This was found in [8] and [10] for a radiation-dominated universe, where  $N = -\frac{2\zeta}{\omega}\rho_m a^3 > 0$  since  $\zeta < 0$  in the magnetic energy dominated theories considered by BSBM. It is important to notice that in universes with an entropy to baryon ratio ( $S \sim 10^9$ ) like our own,  $\psi$  does not experience any growth in time [7]. The constant term on the right-hand side of (40) dominates the solution for  $\psi(\eta)$  throughout the radiation era. Numerical solutions confirm this freezing in of  $\psi$ , and hence of  $\alpha$ , during the radiation era, [6].

#### 4.1.1 Large-scale perturbations in a radiation-dominated era

In the long-wavelength limit ( $w\eta \ll 1$ ) where the scale of the perturbation exceeds the Hubble radius, we can neglect spatial gradients. So, in this limit,  $\frac{\delta\alpha}{\alpha}$  becomes:

$$\delta\psi = \frac{1}{2} \frac{\delta\alpha}{\alpha} \propto \frac{1}{2\eta} e^{i\mathbf{k}\mathbf{x}} \nabla^2 C_2 - 4\pi G w e^{i\mathbf{k}\mathbf{x}} C_2 \eta^{-3}$$

From this expression we can see that on large scales the inhomogeneous perturbations in  $\alpha$  will decrease as a power-law in time. This behaviour agrees with the one found in [9] by other methods.

#### 4.1.2 Small-scale perturbations in a radiation-dominated era

On scales smaller than the Hubble radius ( $w\eta \gg 1$ ) the dominant terms are proportional to  $\nabla^2 C_1$  and  $\nabla^2 C_2$ , so the asymptotic behaviour for  $\frac{\delta\alpha}{\alpha}$  will be:

$$\begin{aligned} \delta\psi &= \frac{1}{2} \frac{\delta\alpha}{\alpha} \propto -\frac{1}{2} w [\cos(\eta w) \nabla^2 C_1 + \sin(\eta w) \nabla^2 C_2] e^{i\mathbf{k}\mathbf{x}} + \\ &+ \frac{1}{2\eta} [\sin(\eta w) \nabla^2 C_1 - \cos(\eta w) \nabla^2 C_2] e^{i\mathbf{k}\mathbf{x}} \end{aligned} \quad (41)$$

On small scales we can see the perturbations on  $\alpha$  will be oscillatory. This behaviour is new and does not coincide with the ones found in [9].

## 4.2 Dust-Dominated Universes

In the case of a flat dust-dominated universe, filled with a  $p = 0$  fluid, we can assume  $\rho_r = \rho_\Lambda = 0 = \kappa$  and  $\delta\rho_r = 0$ . Again, in order to obtain an explicit expression of  $\delta\psi$  from equation (37), we will assume that the matter-field perturbations,  $\rho_m$  and  $\delta\rho_m$ , and the metric perturbation,  $\Phi$ , are not affected by the  $\psi$  perturbations to leading order. Thus, we will assume that these functions behave as in a perturbed flat FRW dust universe.

In general, for a flat dust universe we have the following background solutions:

$$\rho_m = \rho_{m0} a^{-3}, \quad a = \frac{2\pi G \rho_{m0}}{3} \eta^2,$$

where  $\rho_{m0}$  is constant.

As before, we can calculate the most general gauge-invariant solutions for the energy-momentum perturbations  $\delta\rho_m$ , and for the potential  $\Phi$ , assuming that the  $\psi$  field does not affect them to leading order, so their time dependences are [38]:

$$\Phi = C_1 + C_2 \eta^{-5} \quad (42)$$

and

$$\frac{\delta\rho_m}{\rho_m} = \frac{1}{6} \left[ (\eta^2 \nabla^2 C_1 - 12C_1) + (\eta^2 \nabla^2 C_2 + 18C_2) \eta^{-5} \right], \quad (43)$$

where  $C_1$  and  $C_2$  are arbitrary functions of the spatial coordinates.

Once again, we need the background solution for  $\psi$ , in order to calculate  $\delta\psi$  as an explicit function of the conformal time. We use the asymptotic solution

$$\psi = \frac{1}{2} \log \left[ 2N \log \left( \frac{a_m}{6} \eta^3 \right) \right],$$

which was found in [9], as an asymptotic approximation for a dust-dominated universe which is in good agreement with numerical solutions, and where, as above,  $N = -\frac{2\zeta}{\omega} \rho_m a^3 > 0$  is a constant.

### 4.2.1 Large-scale perturbations in a dust-dominated era

On scales larger than the Hubble radius we can neglect the terms proportional to  $\partial_i C_1, \nabla^2 C_1, \nabla^2 C_2$  and  $\partial_i C_2$ ; so in this limit we have the following asymptotic behaviour for the non-decaying mode:



$$\delta\psi = \frac{1}{2} \frac{\delta\alpha}{\alpha} \propto -2C_1 - \frac{2\pi G\rho_{m0}}{\ln(\eta)} C_1 \quad (44)$$

Therefore, on large scales the inhomogeneous perturbations of  $\alpha$  will not grow in time by gravitational instability. This behaviour can be understood with reference to the general evolution equation for  $\psi$  in Friedmann universes. On large scales, where spatial gradients in  $\psi$  can be neglected with respect to its time derivatives, we may view inhomogeneities in density and in  $\psi$  as if they are separate Friedmann universes of non-zero curvature ( $\kappa \neq 0$ ). The growth of inhomogeneity can be deduced by comparing the evolution of  $\alpha$  in the  $\kappa \neq 0$  universes with those in the  $\kappa = 0$  model (a more detailed numerical study of this model will be presented in section 5 below). In effect, this uses the Birkhoff-Newton property of gravitational fields with spherical symmetry. We note that the (18) evolution equation has the simple property that  $\psi$  cannot have a maximum because  $\ddot{\psi} > 0$  when  $\dot{\psi} > 0$  because  $N > 0$ , [9]. This result holds irrespective of the value of  $\kappa \lesseqgtr 0$ . Thus  $\psi$  and  $\alpha$  will continue their slow increase in both over-densities, under-densities and the flat background until we reach scales small enough for spatial derivative to come significantly into play. This has the important consequence that we do not expect large spatial inhomogeneities in  $\alpha$  to have developed. However, it should be noted that the sensitivity of the observations of varying- $\alpha$  effects in quasar spectra is sufficient to discern variations in redshift space smaller than  $O(10^{-5})$ , which is of the same order as the amplitude of density fluctuation on very large scales in the universe.

#### 4.2.2 Small-scale perturbations in a dust-dominated era

On scales smaller than the Hubble radius the dominant terms are those proportional to  $\nabla^2 C_1$  and  $\nabla^2 C_2$ , so the asymptotic behaviour for the growing mode will be:

$$\delta\psi \propto \frac{1}{12} \eta^2 \nabla^2 C_1 \quad (45)$$

This also shows that perturbations of  $\alpha$  will grow on small scales. This result is a product of the assumption that on small scales the universe can be considered as being filled by an homogeneous and isotropic fluid, however we know this is not true below the scale where gravitational clustering becomes

non-linear. On these small scales we also have to worry about new consequences of inhomogeneity which have not been included in our analysis. For example, the constant parameter  $N = -\frac{2\zeta}{\omega}\rho_m a^3 \propto \zeta\Omega_m$  will vary in space due to inhomogeneity in the background matter density parameter  $\Omega_m$  and in the dark matter parameter  $-1 \leq \zeta \leq 1$ . We have assumed that  $\zeta < 0$  for the cold dark matter on large scales in order for the cosmological consequences of time-varying  $\alpha$  to be a small perturbation to the standard cosmological dynamics. But on small scales the dark matter will be baryonic in nature and so  $\zeta > 0$  there. Hence, we expect  $\zeta$  to be significantly scale dependent as we go to small scales. This behaviour will be investigated elsewhere along with the problem of the clustering of inhomogeneities in  $\rho$  and  $\alpha$ .

### 4.3 Accelerated Expansion

In an era of accelerated expansion,  $\ddot{a} > 0$ , as would arise during inflation or during a  $\Lambda$ - or quintessence-dominated epoch at late times, we can consider the scale factor to evolve as a power-law of the conformal time as  $a = \eta^{-n}$ , where  $n \geq 1$  and  $\eta$  runs from  $-\infty$  to 0. The case of  $a = \eta^{-1}$  corresponds to a  $\Lambda$ -dominated epoch.

As in the previous sections, we will assume that all the other matter components which fill the universe will behave exactly as in a perturbed FRW universe with no variations in  $\alpha$ . We also assume that neither of the dust and radiation perturbations will affect the behaviour and evolution of  $\delta\psi$ , and that these perturbations are negligible with respect to the  $\Lambda$  stress driving the expansion, so we will consider  $\delta\rho_m = 0$  and  $\delta\rho_r = 0$ .

It was found in [6] and [10] that in universe which is undergoing accelerated expansion, the asymptotic solution for  $\psi$  is a constant, so we will assume that  $\psi = \psi_\infty$ , in the background, where  $\psi_\infty$  is a constant. Thus, equations (30) and (32) become:

$$\frac{8G\pi|\zeta|\delta\psi\rho_m}{e^{2\psi_\infty}\eta^{2n}} + \nabla^2\Phi + \frac{3n(\eta\Phi - n\Phi')}{\eta^2} = 0$$

$$n(2+n)\Phi + \eta(\eta\Phi'' - 3n\Phi') = 0$$

where we have also considered  $\rho_r = 0$ , but  $\rho_m \neq 0$  because of the coupling with  $\psi$  in the equation of motion of the scalar field. Note that if we had also set  $\rho_m = 0$  here, we would have imposed a no  $\alpha$ -variation condition:  $\delta\psi = 0$ .

Integrating the last equation, we obtain

$$\Phi = \eta^{\frac{3n - \sqrt{1+n(5n-2)}}{2}} \left( \sqrt{\eta} C_1 + \eta^{\frac{1}{2} + \sqrt{1+n(5n-2)}} C_2 \right)$$

where  $C_1$  and  $C_2$  are arbitrary functions of the spatial coordinates. Note that since  $n > 1$  in accelerating universes we see that  $\Phi$  will **decay in time** as  $\eta \rightarrow 0$ . From this solution for  $\Phi$ , we obtain:

$$\begin{aligned} \delta\psi = \frac{1}{2} \frac{\delta\alpha}{\alpha} = & - \frac{e^{2\psi_\infty}}{16G\rho_{m_0}\pi|\zeta|} \eta^{\frac{-3+n-\sqrt{1+n(-2+5n)}}{2}} \left[ 3n \left( 1 + n - \sqrt{1+n(5n-2)} \right) C_1 + \right. \\ & \left. 3n \left( 1 + n + \sqrt{1+n(5n-2)} \right) \eta^{\sqrt{1+n(5n-2)}} C_2 + 2\eta^2 \nabla^2 C_1 + 2\eta^{2+\sqrt{1+n(5n-2)}} \nabla^2 C_2 \right] \end{aligned} \quad (46)$$

which is a decaying function of the conformal time when  $n > 1$  and a constant when  $n = 1$ . Thus, in accord with the expectations of the cosmic no hair theorem, the universe approaches the FRW model and  $\alpha$  is asymptotically constant at late times.

#### 4.3.1 Large-scale perturbations during accelerated expansion

On scales larger than the Hubble radius we can neglect the terms proportional to the spatial derivatives; so in this limit we have the following asymptotic behaviour:

$$\begin{aligned} \delta\psi \propto & - \frac{3e^{2\psi_\infty} n}{16G\rho_{m_0}\pi|\zeta|} \eta^{\frac{n-3-\sqrt{1+n(5n-2)}}{2}} \left[ \left( 1 + n - \sqrt{1+n(5n-2)} \right) C_1 \right. \\ & \left. + \left( 1 + n + \sqrt{1+n(5n-2)} \right) \eta^{\sqrt{1+n(5n-2)}} C_2 \right] \end{aligned} \quad (47)$$

Therefore, as expected, on large scales during an accelerated era inhomogeneities in  $\alpha$  will decrease on time when  $n > 1$  and will be a constant when  $n = 1$ .

#### 4.3.2 Small-scale perturbations during accelerated expansion

On scales smaller than the Hubble radius the dominant terms are the ones proportional to  $\nabla^2 C_1$  and  $\nabla^2 C_2$ , so the asymptotic behaviour will be

$$\delta\psi \propto -\frac{e^{2\psi_\infty}}{8G\rho_{m0}\pi|\zeta|}\eta^{\frac{1+n-\sqrt{1+n(5n-2)}}{2}}\left[\nabla^2 C_1 + \eta^{\sqrt{1+n(5n-2)}}\nabla^2 C_2\right] \quad (48)$$

This confirms that perturbations of  $\alpha$ , as  $\eta \rightarrow 0$ , will decrease on small scales when  $n > 1$  and will be constant when  $n = 1$ .

#### 4.4 Summary of behaviour

In Table 1 we summarise the time evolution of small inhomogeneities in  $\alpha$  found under different conditions in this section.

Universal equation of state	Time Evolution of the perturbations $\delta\psi = \frac{1}{2}\frac{\delta\alpha}{\alpha}$	
	<i>Large scales</i>	<i>Small scales</i>
<i>Radiation-dominated epoch</i> $p = \frac{1}{3}\rho, \quad a \propto \eta$	Decaying	Oscillatory
<i>Dust-dominated epoch</i> $p = 0, \quad a \propto \eta^2$	Constant	Growing
<i>Accelerated expn</i> $a \propto \eta^{-n}, n \geq 1$ $\Lambda$ -dominated $p = -\rho, \quad a \propto \eta^{-1}$ Power-law acceleration, $n > 1$ $p = w\rho, \quad w < 0, \quad a \propto \eta^{-n}$	Constant	Constant
	Decaying	Decaying

Table 1: Time evolution of small inhomogeneities in  $\alpha$ .

## 5 The Non-Linear Regime

In order to study the evolution of inhomogeneities in  $\alpha$  beyond the domain of linear perturbation theory we need to use a different model. The simplest approach is to confine attention to spherically symmetric inhomogeneities. This will be done by comparing the solution of the BSBM theory for  $\alpha$  in a closed ( $\kappa = 1$ ) universe, with the solution for  $\alpha$  in a flat ( $\kappa = 0$ ) universe. We are assuming a Birkhoff property for the BSBM theory so that we can

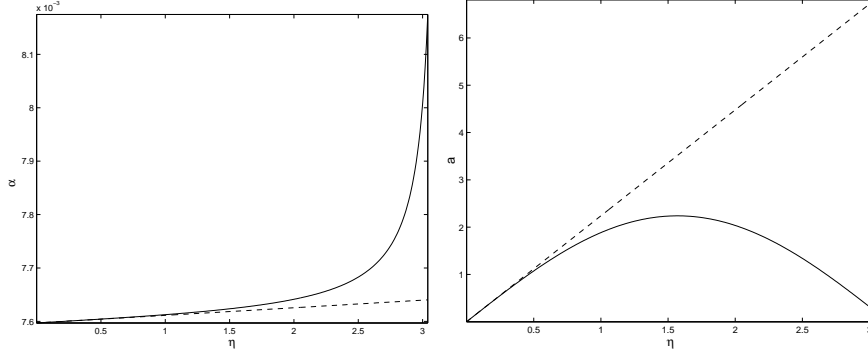


Figure 1: *The evolution of  $\alpha(\eta)$  and  $a(\eta)$  for radiation-dominated universes with  $\kappa = 0$  (dashed) and  $\kappa = 1$  (solid).*

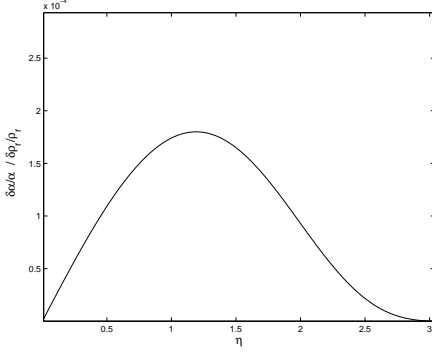


Figure 2: *The evolution of  $\frac{\delta\alpha}{\alpha} / \frac{\delta\rho_r}{\rho_r}$  vs  $\eta$  in radiation-dominated universes.*

treat the perturbation as an independent closed universe. This is a standard technique in general relativity which was first used by Lemaître [33].

We define the alpha 'over-density perturbation' (which is not necessarily small) by

$$\frac{\delta\alpha}{\alpha} \equiv \frac{\alpha_{\kappa=1} - \alpha_{\kappa=0}}{\alpha_{\kappa=0}}$$

where  $\alpha_\kappa$  is the solution of equation (16) for a universe with curvature  $\kappa$ .

## 5.1 Radiation-dominated Era

The scale factor for a radiation-dominated closed ( $\kappa = 1$ ) FRW universe is given by  $a = \sin(\eta)$ ; for a flat ( $\kappa = 0$ ) FRW universe the normalised scale

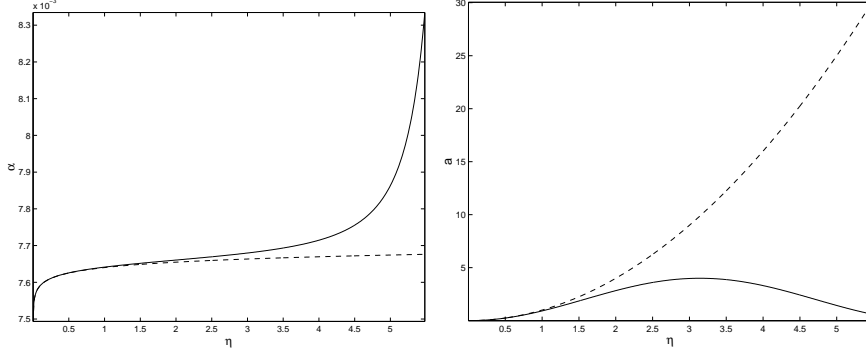


Figure 3: *The evolution of  $\alpha(\eta)$  and  $a(\eta)$  for dust-dominated universes with  $\kappa = 0$  (dashed) and  $\kappa = 1$  (solid).*

factor is given by  $a = \eta$ .

The evolution of  $\alpha$  can be seen in Figure 1 along with the evolution of  $a$  for  $\kappa = 1, 0$ . As expected, we can see there is no difference in the evolution of  $\alpha$  at early times and  $\alpha \propto \eta$  as it was found in [10]. When the difference between the scale factors of the two universes becomes significant, the behaviour of  $\alpha_{\kappa=1}$  begins to deviate from that of  $\alpha_{\kappa=0}$ . We see that  $\alpha_{\kappa=1}$  clearly grows faster than  $\alpha_{\kappa=0}$ . The difference in the growth rates become very significant near the expansion maximum of the bound region ( $\eta \approx \pi$ ). However, after this time our assumption that the background is not affected by changes of  $\psi$  in the cosmological equations that describe the background universe breaks down, since the kinetic energy of the scalar field will diverge and can no longer be neglected in the Friedmann equation. We expect the behaviour near the final singularity to be similar to the kinetic-dominated evolution near the initial singularity discussed in ref. [7].

From Figure 2 we see that the variations in  $\alpha$  will become increasingly important as  $\eta$  approaches the second singularity. Notice that although there is a considerable growth in the perturbations in  $\alpha$ , when we compare them with the perturbations in the radiation, they are not as significant as can be observed from the evolution of the ratio  $\frac{\delta\alpha}{\alpha} / \frac{\delta\rho_r}{\rho_r}$  versus  $\eta$  in Figure 2.

## 5.2 Dust-dominated Era

During the dust-dominated phase of a closed universe ( $\kappa = 1$ ) the normalised scale factor is given by  $a = 2(1 - \cos(\eta))$ , while for a flat universe ( $\kappa = 0$ )

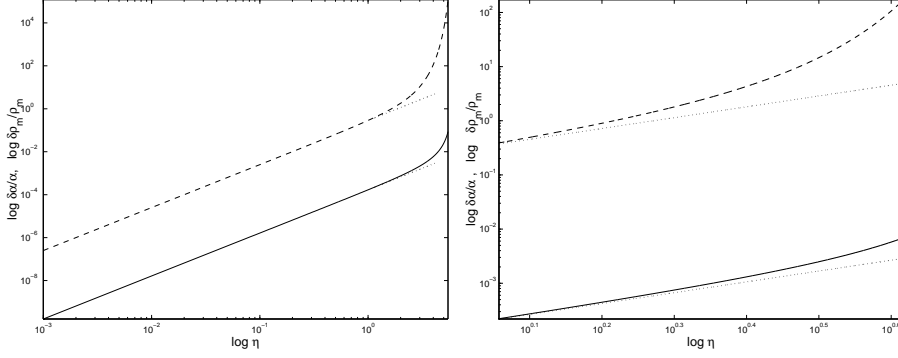


Figure 4: *The evolution of  $\log \frac{\delta\alpha}{\alpha}$  (solid) and  $\log \frac{\delta\rho_m}{\rho_m}$  (dashed) vs  $\log(\eta)$  for a dust-dominated universe. The dotted lines correspond to evolution in the linear perturbation solution.*

the normalised scale factor is given by  $a = \eta^2$ . Integrating equation (16) for both cases we obtain the evolution of  $\alpha$  for both cases.

The evolution of  $\alpha(\eta)$  can be seen in Figure 3 along with the evolution of  $a(\eta)$  for  $\kappa = 1, 0$ . As in the radiation case, we can see there is little difference in the evolution of  $\alpha$  at early times, since the scale factor for the closed model evolves very similarly to the flat one for  $\eta \ll 1$  and  $\alpha \propto \ln(\eta)$ . The differences between  $\alpha_{\kappa=1}$  and  $\alpha_{\kappa=0}$  cases start to appear when  $\eta \approx 1$ , when the nonlinear regime commences. These differences become more accentuated near the second singularity, but once again this is the region where our approximations break down.

Notice that although there is a considerable growth in the perturbations in  $\alpha$ , when we compare them with the perturbations in the cold dark matter, they are not as significant as can be observed from the evolution of  $\log(\frac{\delta\alpha}{\alpha})$  and  $\log(\frac{\delta\rho_m}{\rho_m})$  vs.  $\log(\eta)$  in Figure 4. In this case perturbations in  $\alpha$  are even less significant than in the radiation case. We note that the fact that the linear regime is a very good approximation at early times, since as can be seen from Figure 4,  $\frac{\delta\alpha}{\alpha}$  tracks  $\frac{\delta\rho_m}{\rho_m} \propto t^{2/3}$  at early times. From the detail of Figure 4 we see that variations in the cold dark matter will start to occur before than variations in  $\alpha$ , and  $\frac{\delta\rho_m}{\rho_m}$  is at least three orders of magnitude bigger than  $\frac{\delta\alpha}{\alpha}$ .

## 6 Conclusions

By applying the gauge-invariant formalism of ref. [38] to the simple BSBM theory of varying  $\alpha$  we have determined the evolution of small inhomogeneities in  $\delta\alpha/\alpha$  in the presence of small adiabatic density inhomogeneities. To leading order, the evolution of the perturbations to the expansion dynamics and the matter and radiation content of the universe behave as in cosmological models with constant  $\alpha$  and we can determine the behaviour of small inhomogeneities in  $\delta\alpha/\alpha$  in the gravitational fields created by the density and metric perturbations (see also refs. [9] and [10]).

In a flat radiation-dominated universe we find that inhomogeneous perturbations in  $\alpha$  will decrease on large scales while on scales smaller than the Hubble radius they will undergo bounded oscillations. In reality, we expect dissipation of the adiabatic fluctuations to occur by Silk damping and small-scale fluctuations in  $\alpha$  will also undergo decay as their driving terms damp out. However, while the exact solution for the evolution of  $\alpha$  is a linear sum of a constant and a slow power-law growth the power-law evolution does not become dominant by the end of the radiation era in universes like our own with entropy per baryon  $O(10^9)$ .

In a flat dust-dominated universe small inhomogeneities in  $\alpha$  will become constant on large scales at late times while on small scales they will increase as  $t^{2/3}$ . In reality, the small-scale evolution will be made more complicated by the breakdown of the assumptions underlying the perturbation analysis and the development of local deviations from the FRW behaviour.

In an accelerated phase of our universe, as is the case for an early inflationary epoch, or during a  $\Lambda$ - or quintessence-dominated late phase of evolution, we show that inhomogeneous perturbations in  $\alpha$  will decrease on all scales. This result complements the earlier discovery [6], [7], that  $\alpha$  tends to a constant with exponential rapidity in Friedmann universes that become dominated by a cosmological vacuum stress. Any pre-existing inhomogeneities will be frozen in but their scale will be exponentially increased by the de Sitter expansion.

These perturbative results are quite good approximations when we consider large scales, but are expected to break down when extended to small scales, where non-linear effects come into play and local deviations from isotropy and homogeneity of the matter content are significant. We note that the background solutions for  $\psi$ , about which we have linearised the perturbations of the Einstein equations, are solutions which describe the time



evolution of  $\psi$  on a 'standard' FRW background. Our neglect of the back-reaction of the  $\psi$  perturbations on the background expansion dynamics is a good approximation up to logarithmic corrections.

When we examined the non-linear evolution of spherical inhomogeneities by means of a comparative numerical study of flat and closed Friedmann models we found that perturbations in  $\alpha$  remain almost negligible with respect to perturbations in the fluid that dominates the energy density of the universe at the same epoch. Comparing the flat with the closed solution for  $\alpha$ , we concluded that in both the cases of a radiation or dust-dominated epoch,  $\alpha$  will 'feel' the change of behaviour in the scale factor, and the perturbations  $\frac{\delta\alpha}{\alpha}$  will grow in time. The early-time behaviour of the non-linear solutions confirms the linear behaviour found in section 4. In particular  $\frac{\delta\alpha}{\alpha}$  changes in proportion to  $\frac{\delta\rho}{\rho}$  at early times.

We have provided a detailed analysis of the behaviour of inhomogeneous perturbations in  $\alpha$  and its time variation on large scales under the assumption that the defining constant of the BSBM theory,  $\zeta$ , is constant and negative in sign. In reality this assumption will break down on small scales. The negativity of the effective value of  $\zeta$  requires that the cold dark matter is dominated by the magnetic rather than the electric field energy (see also the discussion of ref [6] and by Bekenstein [40]). However, on sufficiently small scales the dark matter will become dominated by baryons and the sign of the effective  $\zeta$  will have to change sign. Overall, there will also be a gradient in the value of  $|\zeta|$  reflecting the scale dependence of the relative contribution of dark matter to the total density of the universe. We have not included these effects in the present analysis. They would need to be included in any detailed analysis of the small-scale behaviour of inhomogeneities in  $\alpha$ . This is an important challenge for future work because it would enable the quasar data on varying  $\alpha$  to be compared directly with the limits from the Oklo natural reactor [34, 32] and Rhenium-Osmium abundances in meteorites [36, 37]. At present the relation between the cosmological and geonuclear evidences is unclear because the latter are derived from physical processes occurring within the cosmologically non-evolving solar system environment.

Variations in  $\alpha$  also affect the cosmic microwave background radiation spectrum and anisotropy in different ways, but the effects must be disentangled from allowed changes in other cosmological parameters that can contribute similar effects. These changes in the microwave background with  $\alpha$  left as a free constant parameter were analysed in [41] using the new WMAP

data. They are far less sensitive than the many-multiplet analyses of quasars at  $z = 0.5-3$  [1, 2, 3, 4], although they derive from higher redshifts,  $z < 1100$ . These studies can accommodate constant and varying  $\alpha$  but up to a level that would be too large to be consistent with the quasar data and the slow time-evolution of the theory with time-varying  $\alpha$  described in this paper.

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